MATHEMATICS

Time Allowed : 3 hours Full Marks : 100

The figures in the margin indicate full marks for the questions.

Answer any 10 (ten) questions taking at least 3 (three) questions from each section.

SECTION A

1. (a) Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix},$$ find a matrix P such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ **(10)**

(b) In $V_3(R)$, where R is the field of real numbers, examine each of the following sets of vectors for linear dependence.

(i) $\{(1,2,0),(0,3,1),(-1,0,1)\}$ **(5)**

(ii) $\{(2,3,5),(4,9,25)\}$ **(5)**

2. (a) Define Taylor's Theorem in Lagrange's form of remainder. Hnece deduce that if $f(x)=(1+x)^4$ then $(1+x)^4 = 1+4x+6x^2+4x^3+x^4$ **(5)**

(b) If $u = f(x,y)$ and $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=\frac{\delta^2 u}{\delta r^2}+\frac{1}{r}.\frac{\delta u}{\delta r}+\frac{1}{r^2}.\frac{\delta^2 u}{\delta \theta^2}$$ **(5)**

3. (a) Evaluate: **(5)**

(i) $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$

(ii) $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}dx$

(Contd. 3)

(b) Evaluate:

$\iiint xyz \sin(x+y+z)\, dxdydz$ the integral being extended to all positive values of the variables subject to the condition, $x+y+z \leq \frac{\pi}{2}$ **(5)**

4. (a) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular generation, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ **(5)**

(b) Find the equation of the tangent plane to the paraboloid $ax^2+by^2=bz$ parallel to the plane $lx+my+nz=0$ **(5)**

5. Solve <u>any two</u> differential equations: **(10)**

(a) $\left(1+x^2\right)^2 \frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}+4y=0$

(b) $y_2+x^2y=\sec nx$ by method of variation of parameter.

(c) $\sqrt{x}\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+3y=x$

6. (a) A point moving in a straight line with uniform acceleration describes a and b feet in successive interval of t_1 and t_2 seconds. Prove that the acceleration is $\dfrac{2(bt_1 - at_2)}{t_1 t_2 (t_1 + t_2)}$

Also prove that if the point describes successive equal distances in time t_1, t_2, t_3 then

$$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}$$ **(5)**

(b) A heavy uniform chain of length $2l$, hangs over a small smooth fixed pulley, the length $l + c$ being at one side and l - c at the other. If the end of the shorter portion be held and then let go, show that the chain will slip off the pulley in time

$$\sqrt{\frac{l}{g}}.\log\frac{l + \sqrt{l^2 - c^2}}{c}.$$ **(5)**

7. (a) A heavy solid right circular cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased, determine whether the initial motion of the cone will be one of sliding or tumbling over. **(5)**

(b) Find the power of an engine that would empty in 44 minutes a cylindrical well full of water, if the diameter of the wall is 3.5 m, its depth 16 m, and if water is raised by pumping to a level ground 22 m above the surface of the well. **(5)**

(Contd. 5)

8. (a) Prove that the depth of the centre of the pressure of a trapezium immersed in water with the side *a* in the surface and the parallel side *b* at a depth *h* below the surface is $\left(\frac{a+3b}{a+2b}\right)\cdot\frac{h}{2}$ **(5)**

(b) A hollow cone filled with water and closed, is held with its axes horizontal, find the resultant vertical pressure on the upper half of its curved surface. **(5)**

9. (a) If $f = x^2zi - 2y^3z^2j + xy^2zk$, find $divf$, $curlf$, at $(1,-1,1)$ **(5)**

(b) Verify Stoke's theorem for
$F = (2x-y)i - yz^2j - y^2zk$
where *s* is upper half surface of the sphere
$x^2 + y^2 + z^2 = 1$ and *C* is its boundary. **(5)**

SECTION B

10. (a) Let $\{G,0\}$ be a semi group and for any two elements a, b in G, each of the equation $a_0x = b$ and $y_0a = b$ has a solution in G. Then $\{G,0\}$ is a group. **(5)**

(b) Let $\{R,+,\cdot\}$ be a ring. A non-empty subset s of R forms a subring of R if and only if

(i) $\{s,+\}$ is a sub group of $\{R,+\}$ and

(ii) $a \in S$, $b \in s \Rightarrow a.b \in s$ **(5)**

11. (a) If f is bounded and integrable on $[a,x]$ for all $x \geq a$ and if $\int_0^\infty |f(x)|dx$ exists, then $\int_0^\infty f(x)dx$ also exists.

In other words, the integral $\int_0^\infty f(x)dx$ will converge if it converges absolutely. **(5)**

(b) Determine the radii of convergence of the following power series. **(5)**

(i) $\sum_{n=0}^{\infty} \frac{2n!}{(n!)^2}.z^n$

(ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}.z^n$

12. Solve the following linear programming problem by the revised simplex method: **(10)**

$3x_1 + 4x_2 \leq 6$

$6x_1 + x_2 \leq 3$, $x_1 x_2 \geq 0$

$Max_z = 2x_1 + x_2$

13. (a) Solve:

(i) $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$ **(5)**

(ii) $\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)}$ **(5)**

(b) Find complete integral of:

$p^2 + q^2 - 2px - 2qy + 1 = 0$ **(5)**

14. (a) Using Newton-Raphson method, find correct to four decimals the root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$ **(5)**

(b) Using Runge-Kutta method, find an approximate value of y for $x = 0.2$, if $\frac{dy}{dx} = x + y^2$ given $y = 1$ when $x = 0$ **(5)**

15. A sphere of radius *R*, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density r, which is at rest at infinity. If the pressure at infinity is p, show that the pressure at the surface of the sphere at time

t is $\pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$

If $R = a(2 + \cos nt)$, show that, to prevent cavitation in the fluid, π must not be less than $3\rho a^2 n^2$. **(10)**

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