CSM : 15

MATHEMATICS
PAPER - I

Time Allowed : 3 hours | Full Marks : 100

Marks for each question is indicated against it.

Attempt any 5 (five) questions taking not more than 3 (three) questions from each Part.

PART A

1. (a) Let $A \in M_n(F)$ and let $\lambda_1, \lambda_2, ...\lambda_m \in F$ be distinct characteristic roots of A.

If $\{v_1, v_2, v_3 ....v_m\}$ are the corresponding characteristic vectors.

Prove that $\{v_1, v_2, v_3 ....v_m\}$ are linearly independent. **(10)**

(b) Let W be any subspace of a finite dimensional vector space V. Show that there exists a subspace W' of V such that $V = W + W'$ and $W \cap W' = \{0\}$. Further show that for any such subspace W', $\dim V = \dim W + \dim W'$. **(10)**

2. (a) Prove that the function f, where

$$f(x,y) == \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$ **(5)**

(b) Find all the maxima and minima of the function given by

$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy.$ **(5)**

(c) If $u = \dfrac{x^2+y^2+z^2}{x}$, $v = \dfrac{x^2+y^2+z^2}{y}$, $w = \dfrac{x^2+y^2+z^2}{z}$, then show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{x^2y^2z^2}{\left(x^2+y^2+z^2\right)^3}$$ **(10)**

3. (a) Prove that $\int \frac{(x^4-1)dx}{x^2\sqrt{x^4+x^2+1}} = \frac{\sqrt{x^4+x^2+1}}{x} + c$ **(5)**

(b) Prove that $\iint_R xy\,dA = \frac{243}{8}$,

where R is the region enclosed by the curves $y = x^2$ and $y = 3x$. **(5)**

(c) Prove that $\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \frac{\pi}{2}$ **(5)**

(d) Evaluate the integral $\iint \sqrt{4x^2 - y^2}\,dxdy$ over the triangle formed by the straight lines $y = 0,\ x = 1,\ y = x.$ **(5)**

4. (a) Find the shortest distance between the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

and the equations of the line of shortest distance **(7)**

(b) Show that the equation $2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$ represents an elliptic paraboloid. **(6)**

(c) Prove that the planes $lx + my + p = 0$ and $l'x + m'y + p' = 0$ are conjugate diametral planes of the paraboloid $ax^2 + by^2 = 2cz$ if $\frac{ll'}{a} + \frac{mm'}{b} = 0.$ **(7)**

PART B

5. (a) Apply the method of variation of parameters to solve the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$$ **(10)**

(b) Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ is a parameter. **(6)**

(c) Solve: $y = px + a\sqrt{1+p^2}$ **(4)**

6. (a) Establish the formula $\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}$ for the motion of a particle describing a central orbit under an attractive force P per unit mass, the symbols having usual meaning. **(10)**

(b) A body is projected at an angle α to the horizon, so as just to clear two walls of equal height a at a distance $2a$ from each other.

Show that the range is equal to $2a\cot\frac{\alpha}{2}$. **(5)**

(c) A body of mass (m_1+m_2) moving in a straight line is split into two parts of masses m_1 and m_2 by an external explosion which generates kinetic energy E. If after the explosion, the two parts move in the same line as before, show that their relative velocity is $\sqrt{\frac{2E(m_1+m_2)}{m_1m_2}}$ **(5)**

7. (a) A rod AB is movable about a point A and to the point B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is $\frac{w\cos\alpha\cos\beta}{2\sin(\alpha+\beta)}$ **(10)**

(b) The least force which will move a weight up an inclined plane is P. Show that the least force acting parallel to the plane which will move the weight upwards is $P\sqrt{1+\mu^2}$, where μ is the coefficient of friction. **(10)**

8. (a) Show that $curl\ curl\vec{f}=\vec{\nabla}div\vec{f}-\nabla^2\vec{f}$. **(10)**

(b) Evaluate $\oint_\Gamma (e^x dx+2ydy-dz)$ by using Stokes' theorem

where Γ is the curve $x^2+y^2=1,\ z=2$. **(6)**

(c) Prove that the necessary and sufficient condition that a proper vector $\vec{u}(t)$ has a constant magnitude is that $\vec{u}(t).\frac{d\vec{u}}{dt}(t)=0$. **(4)**

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