CSM : 15

MATHEMATICS

PAPER - I

Time Allowed : 3 hours
Full Marks : 100

Marks for each question is indicated against it.

Attempt any 5 (five) questions taking not more than 3 (three) questions from each Part.

PART A

1. (a) Let $A \in M_n(F)$ and let $\lambda_1, \lambda_2, \ldots, \lambda_m \in F$ be distinct characteristic roots of $A$. If $\{v_1, v_2, v_3, \ldots, v_m\}$ are the corresponding characteristic vectors. Prove that $\{v_1, v_2, v_3, \ldots, v_m\}$ are linearly independent. (10)

(b) Let $W$ be any subspace of a finite dimensional vector space $V$. Show that there exists a subspace $W'$ of $V$ such that $V = W + W'$ and $W \cap W' = \{0\}$. Further show that for any such subspace $W'$, $\dim V = \dim W + \dim W'$. (10)

2. (a) Prove that the function $f$, where

$$
f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0)
\end{cases}
$$

is continuous at $(0, 0)$ (5)

(b) Find all the maxima and minima of the function given by

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$ (5)

(c) If $u = \frac{x^2 + y^2 + z^2}{x}$, $v = \frac{x^2 + y^2 + z^2}{y}$, $w = \frac{x^2 + y^2 + z^2}{z}$, then show that

$$\frac{\partial}{\partial (u, v, w)} \left( \frac{x^2 y^2 z^2}{(x^2 + y^2 + z^2)^3} \right) = \frac{x^2 y^2 z^2}{(x^2 + y^2 + z^2)^3}$$ (10)
3. (a) Prove that \[
\int \frac{(x^4 - 1)\, dx}{x^2\sqrt{x^4 + x^2 + 1}} = \frac{\sqrt{x^4 + x^2 + 1}}{x} + c \quad (5)
\]

(b) Prove that \[
\iint_R xy\, dA = \frac{243}{8},
\]
where \(R\) is the region enclosed by the curves \(y = x^2\) and \(y = 3x\). \(5\)

(c) Prove that \[
\int_0^2 \frac{\, dx}{\sqrt{4 - x^2}} = \frac{\pi}{2} \quad (5)
\]

(d) Evaluate the integral \[
\iint (4x^2 - y^2)\, dxdy \quad \text{over the triangle formed by the straight lines} \quad y = 0, \ x = 1, \ y = x. \quad (5)
\]

4. (a) Find the shortest distance between the lines
\[
\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}
\]
and the equations of the line of shortest distance \(7\)

(b) Show that the equation \(2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0\) represents an elliptic paraboloid. \(6\)

(c) Prove that the planes \(lx + my + p = 0\) and \(l'x + m'y + p' = 0\) are conjugate diametral planes of the paraboloid \(ax^2 + by^2 = 2cz\) if \(\frac{l'l'}{a} + \frac{mm'}{b} = 0\). \(7\)

**PART B**

5. (a) Apply the method of variation of parameters to solve the equation
\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x \quad (10)
\]

(b) Find the orthogonal trajectories of \(\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1\), where \(\lambda\) is a parameter. \(6\)

(c) Solve: \(y = px + a\sqrt{1 + p^2}\) \(4\)
6. (a) Establish the formula \[ \frac{d^2u}{d\theta^2} + u = \frac{P}{\hbar^2 u} \] for the motion of a particle describing a central orbit under an attractive force \( P \) per unit mass, the symbols having usual meaning. \((10)\)

(b) A body is projected at an angle \( \alpha \) to the horizon, so as just to clear two walls of equal height \( a \) at a distance \( 2a \) from each other.

Show that the range is equal to \[ 2a \cot \frac{\alpha}{2}. \] \((5)\)

(c) A body of mass \( (m_1 + m_2) \) moving in a straight line is split into two parts of masses \( m_1 \) and \( m_2 \) by an external explosion which generates kinetic energy \( E \). If after the explosion, the two parts move in the same line as before, show that their relative velocity is \[ \sqrt{\frac{2E}{m_1m_2}} \] \((5)\)

7. (a) A rod AB is movable about a point A and to the point B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is \[ \frac{w \cos \alpha \cos \beta}{2 \sin(\alpha + \beta)} \] \((10)\)

(b) The least force which will move a weight up an inclined plane is \( P \). Show that the least force acting parallel to the plane which will move the weight upwards is \[ P \sqrt{1 + \mu^2} \], where \( \mu \) is the coefficient of friction. \((10)\)

8. (a) Show that \[ \text{curl} \text{ curl} \vec{f} = \vec{\nabla} \text{div} \vec{f} - \nabla^2 \vec{f}. \] \((10)\)

(b) Evaluate \[ \int_{\Gamma} (e^x dx + 2ydy - dz) \] by using Stokes’ theorem where \( \Gamma \) is the curve \( x^2 + y^2 = 1, \ z = 2. \) \((6)\)

(c) Prove that the necessary and sufficient condition that a proper vector \( \vec{u}(t) \) has a constant magnitude is that \[ \vec{u}(t) \cdot \frac{d\vec{u}}{dt}(t) = 0. \] \((4)\)

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