CSM : 14

MATHEMATICS

PAPER - I

Time Allowed : 3 hours | Full Marks : 100

Marks for each question is indicated against it.

Attempt *<u>any 5 (five)</u>* *questions taking not more than 3 (three) questions from each Part.*

PART A

1. (a) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 3 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(10)

(b) Let $A = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix} \in M_3(R)$. Find the characteristic polynomial of A. Prove that the minimal polynomial of A is equal to the characteristic polynomial. **(5+5=10)**

2. (a) Show that $-2x^3 + 15x^2 - 36x + 6$ is strictly increasing on $(2,3)$. **(5)**

(b) Prove that $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x}} = 1$ **(5)**

(c) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous but not differentiable at the origin. **(10)**

3. (a) Test the convergence of the following improper integrals:

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ **(5)**

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p} dx$ **(5)**

(b) By using double integration, find the

(i) area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ **(5)**

(ii) area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant . **(5)**

4. (a) Find the equation of a sphere through the intersection of the spheres $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 - 2x + 4y = 3$ and passing through the point $(1,2,-3)$ **(6)**

(b) Find the equation of a right circular cylinder generated by lines parallel to the line $\frac{x-2}{1} = \frac{y+2}{2} = \frac{z-1}{3}$, the guiding curve being the conic $z = 2,\ 3x^2 + y^2 = 1$ **(7)**

(c) Find the equation of the curve in which the plane $z = h$ cuts the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and find the area enclosed by the curve. **(7)**

PART B

5. (a) Find the complete solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$$ **(10)**

(b) Solve the following differential equation by method of variation of parameter:

$$\frac{d^2y}{dx^2} + y = x - \cot x$$ **(10)**

6. (a) A rocket is released at a point A from a jet aircraft flying horizontally at 1000km/hr at an altitude of 800m. If the rocket thrust remains horizontal and gives the rocket a horizontal acceleration of 0.5g, determine the angle θ from the horizontal to the line of sight to the target. **(8)**

(b) A particle moves under a central repulsive force which is equal to $\frac{m\mu}{(\text{distance})^2}$ and is projected from an apse at a distance *a* with velocity *V*. Show that the equation to the path is $r\cos p\theta = a$, and that the angle θ described in time *t* is $\frac{1}{v}\tan^{-}\left(\frac{pV}{a}t\right)$, where $p^2 = \frac{\mu + a^2V^2}{a^2V^2}$. **(12)**

7. (a) The sum of moments of a system of coplanar forces about each of three non-collinear points in the plane is the same (without being equal to zero). Prove that the system is equivalent to a couple. **(7)**

(b) A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, prove that $\tan\theta = 2\mu$, where μ is the coefficient of friction. **(6)**

(c) A heavy uniform rod of length 2*a* rests in equilibrium, having one end against a smooth vertical wall and being placed upon a peg at a distance *b* from the wall. Show that the inclination of the rod to the vertical is $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$. **(7)**

8. If $\vec{r}$ is a vector and $r = |\vec{r}|$, then prove that

(a) $\nabla^2\left(\frac{1}{r}\right) = 0$ or $div\ grad\left(\frac{1}{r}\right) = 0$ **(10)**

(b) $div\left(r^n\vec{r}\right) = (n+3)r^n$ **(10)**

* * * * * * *