CSM : 22

MATHEMATICS

PAPER - I

Time Allowed : 3 hours Full Marks : 100

Marks for each question is indicated against it.
Attempt any 5 (five) questions taking not more than 3 (three) questions from each Part.

PART - A

1. (a) Let A and B be subspaces of a finite dimensional vector space V over a field K. Then, prove that dim(A+B) = dim A + dim B - dim (A $\cap$ B). **(10)**

(b) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find a matrix P such that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. **(10)**

2. (a) Prove that the function f defined by

$$f(x) = \begin{cases} 0 \; for \; x^2 > 1 \\ 1 \; for \; x^2 < 1 \\ \frac{1}{2} \; for \; x^2 = 1 \end{cases}$$

has discontinuities at $x = \pm 1$. **(4)**

(b) State and prove Cauchy's Mean Value Theorem. **(2+6=8)**

(c) Find the asymptotes of $xy^2 - y^2 - x^3 = 0$. **(4)**

(d) Evaluate $\lim_{x \to 0} (\sin x)^{2 \tan x}$. **(4)**

3. (a) Evaluate *any two* of the following: **(3×2=6)**

(i) $\int_0^{\pi/2} \frac{dx}{4 + 5\cos x}$ (ii) $\int \frac{dx}{9x^2 - 12x + 8}$ (iii) $\int \frac{(x-1)}{(x-2)(x-3)} dx$

(b) Evaluate : $\lim_{n \to 0} \left[\frac{(n+1)(n+2)(n+3)..........(n+n)}{n^n} \right]^{1/n}$ **(5)**

(c) Show that $\int_0^{\pi} \log(1 + \cos x)\, dx = \pi \log\left(\frac{1}{2}\right)$ **(5)**

(d) Examine the convergence of $\int_0^{\infty} \frac{dx}{1 + x^2}$ **(4)**

4. (a) Find the shortest distance between the lines

$$\frac{x-3}{-3} = \frac{y-8}{1} = \frac{z-3}{-1} \text{ and } \frac{x+3}{3} = \frac{y+7}{-2} = \frac{z-6}{-4}.$$

Also find the equation of shortest distance. **(6+2=8)**

(b) Find the equation of the plane which passes through the line of intersection of the planes $2x - 3y + 4z = 0$ and $4x + y - 2z = 8$, and perpendicular to the plane $x + y + z = 9$. **(6)**

(c) Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation. **(6)**

PART - B

5. (a) Find the orthogonal trajectories of the family of the curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, where l is the parameter. **(7)**

(b) Solve $y = px + \sqrt{a^2 p^2 + b^2}$ **(6)**

(c) Solve $\frac{d^2 y}{dx^2} + 16y = 1,\ y(0) = 1,\ y'(0) = 2$ **(7)**

6. (a) If a be the amplitude and T be the period of a particle executing S. H. M. in a straight line, show that the time taken by the particle to travel a distance x from the centre of force is $\frac{T}{2\pi}\sin^{-1}\left(\frac{x}{a}\right)$ and the velocity in that position is $\frac{2\pi}{T}\sqrt{a^2 - x^2}$. **(6)**

(b) A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension of the string has values mW and nW respectively, when the particle is at the highest and the lowest points of depth. Show that $n = m + 6$. **(8)**

(c) A ball is projected so as to just clear two walls, the first of height a at a distance b from the point of projection and the second of height b and at a distance a from the point of projection. Show that the range on the horizontal plane is $\frac{a^2 + ab + b^2}{a+b}$ and the angle of projection exceeds $\tan^{-1}3$. **(6)**

7. (a) A string of length a forms the shorter diagonal of a rhombus of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported on a horizontal position, prove that tension of the string is $\dfrac{2W\,(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$. **(10)**

(b) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone, so that equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere. **(10)**

8. (a) Evaluate the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q has co-ordinates $(5, 0, 4)$. **(7)**

(b) Show that the vector $\mathbf{f} = (\sin y + z)\mathbf{i} + (x \cos y - z)\mathbf{j} + (x - y)\mathbf{k}$ is irrotational. **(7)**

(c) Evaluate by Stocke's theorem $\oint_c \left(e^x dx + 2y dy - dz\right)$ where C is the curve $x^2 + y^2 = 4$, $z = 2$. **(6)**

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